On forward improvement iteration for stopping problems

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Abstract. We consider the optimal stopping problem for $g(Z_n)$, where $Z_n, n = 1, 2 \ldots$ is a homogeneous Markov sequence. An algorithm, called forward improvement iteration, is investigated by which an optimal stopping time can be computed. Using an iterative step, this algorithm computes a sequence $B^0 \supseteq B^1 \supseteq B^2 \supseteq \ldots$ of subsets of the state space such that the first entrance time into the intersection F of these sets is an optimal stopping time.
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1 Introduction

We recall the general situation for a problem of optimal stopping. Starting with a probability space and a time set $T \subseteq [0, \infty]$, we have a filtration $(\mathcal{A}_t)_{t \in T}$ and an adapted real-valued stochastic process $(X_t)_{t \in T}$. A stopping rule is a mapping $\tau : \Omega \to T$ satisfying $\{\tau \leq t\} \in \mathcal{A}_t$ for all $t \in T$, and we let $\mathcal{T}$ denote the set of all stopping rules. We assume that EX_τ exists (possibly infinite) for all $\tau \in \mathcal{T}$. The aim is to find a stopping rule τ^* satisfying

$$EX_{\tau^*} = \sup_{\tau \in \mathcal{T}} EX_\tau$$

and to compute this supremum, called the value of the stopping problem. There is a wealth of literature on the theory of optimal stopping and its applications, see e.g. the recent monograph by Peskir and Shiryaev (2006), current research often triggered by the applicability in option pricing problems of mathematical finance.

Differing from the well-known backwards induction approach, the following algorithm called forward improvement iteration (FII) was introduced in Irle (1980) as a general theoretical tool and was then used for finding explicit solutions in best choice problems.

For FII we let $T = \{0, 1, 2, \ldots, m\}$ for $m \leq \infty$ thus including the infinite time case $m = \infty$. For any sequence $\mathcal{C} = (C_n)_{n \in T}$ of $C_n \in \mathcal{A}_n$ with $C_m = \Omega$ we define the stopping rule

$$\tau_n(\mathcal{C})(\omega) = \inf \{k \geq n : \omega \in C_k\}.$$

For $\mathcal{C}$ we define $\mathcal{C}^*$ by $C^*_m = \Omega$ and

$$C^*_n = \{E(X_{\tau_{n+1}(\mathcal{C})} | \mathcal{A}_n) \leq X_n\} \cap C_n, \; n < m.$$

FII proceeds in the following way. Let $\mathcal{C}^0 = (\Omega)_n$ and by induction $\mathcal{C}^k = (\mathcal{C}^{k-1})^*$. Define $\mathcal{D}$ by $D_n = \bigcap_k C^k_n$. It is shown in Irle (1980) that $\tau_0(\mathcal{D})$ is an optimal stopping rule under the condition, trivially true for $m < \infty$, that $EX_{\lim \sigma_n} = \lim EX_{\sigma_n}$ for any increasing sequence $(\sigma_n)_n$ of stopping rules.

This approach was adapted in Irle (2006) to Markovian stopping problems. Different to the general case where FII works in the set of sample paths, the adapted algorithm now works in the space set of a Markov chain which allows for efficient numerical calculations.

We consider $T = \{0, 1, 2, \ldots, \infty\}$ and a homogeneous Markov process $(Z_n)_{n<\infty}$ with respect to the underlying filtration. The measurable state space is denoted by $(S, \mathcal{S})$. Let $g : S \to \mathbb{R}$ be measurable. We look at the optimal stopping problem for

$$X_n = g(Z_n), \;\; n < \infty, \;\; X_\infty = \limsup X_n,$$

formally writing $g(Z_\infty)$ for X_∞ in various expressions. We use P_z, E_z for $P(\cdot|Z_0 = z)$, $E(\cdot|Z_0 = z)$ and assume that $E_z\, g(Z_\tau)$ exists for all stopping rules τ and all $z \in S$. We are looking for a stopping rule τ^* such that for all $z \in S$

$$E_z\, g(Z_{\tau^*}) = \sup_\tau E_z\, g(Z_\tau) = V(z), \text{ say},$$

V referred to as value function. FII works in the following way. For a measurable $B \subseteq S$ set

$$\tau_n(B) = \inf\{j \geq n : Z_j \in B\}$$

and define

$$B^* = \{z : g(z) \geq E_z g(Z_{\tau_1(B)}\} \cap B.$$

Let $B^0 = S$ and by induction $B^k = (B^{k-1})^*$, furthermore

$$F = \bigcap_k B^k.$$

Under the condition $E_z g(Z_{\lim \sigma_n}) = \lim E_z g(Z_{\sigma_n})$ for all z and all increasing sequences $(\sigma_n)_n$ of stopping rules, $\tau_0(F)$ is an optimal stopping rule. The above condition may be omitted in the case of a finite state space S and then the algorithm terminates after at most $|S|$ steps. In the case of discounting with $0 < \alpha \leq 1$ we consider the stopping problem for $X_n = \alpha^n g(Z_n)$. For $\alpha = 1$ this reduces to $X_n = g(Z_n)$, but on the other hand, the discounted case may also be viewed as a non-discounted stopping problem for the space-time chain (Z_n, n). The algorithmic step of going from B to B^* is provided by

$$B^* = \{z : g(Z) \geq E_z \alpha^{\tau_1(B)} g(Z_{\tau_1(B)})\} \cap B.$$

We refer to Irle (2006) where also various examples with discounting were treated. In these examples, the algorithmic step of going from B to B^* was performed by providing numerical values for the quantities $E_z \alpha^{\tau_1(B)} g(Z_{\tau_1(B)})$ by path wise simulations of the Markov chain. As shown theoretically and demonstrated by actual computations, FII finds, for finite state space S, the value function and the optimal stopping time in a finite number of iterations. In the examples of Irle (2006), this number was surprisingly low.

The purpose of this note is twofold. Firstly we shall show that the algorithmic step may be described in terms of a linear equation which allows the use of fast methods of numerical linear algebra and the handling of very large state spaces. Secondly we shall show how the algorithm may be adapted to continuous time Markov chains.

2 The algorithmic step as a linear equation

We consider a discounted Markovian stopping problem as described in the introduction with a finite state space $S, g : S \to \mathbb{R}, 0 < \alpha \leq 1$. We assume $g \geq 0$ for $\alpha < 1$ to rule out that infinite observation is optimal due to discounting. Let

$$p_{zy} = P(Z_1 = y \,|\, Z_0 = z), \;\; y, z \in S.$$

For shorter notation we write $h_i(B)(z) = E_z\, \alpha^{\tau_i(B)} g(Z_{\tau_i(B)}), \; z \in S, \; i = 0, 1.$

Proposition 1 *Let $B \subseteq S$ with $P_z(\tau_0(B) < \infty) = 1$ for all $z \in S$. Then $h_0(B)$ is the unique solution of*

$$h(z) = g(z), \, z \in B, \; h(z) = \alpha \sum_y p_{zy}\, h(y), \; z \in S \setminus B.$$

Proof. Obviously $h_0(B)$ fulfills $h_0(B)(z) = g(z),\ z \in B$, by definition. Furthermore $h_0(B)(z) = \sum_y p_{zy}\, \alpha\, h_0(B)(y),\ z \in S \setminus B$, as a well-known consequence of the Markov property which is often denoted as the α-harmonicity of $h_0(B)$ on $S \setminus B$. So $h_0(B)$ provides a solution. Using the condition $P_z(\tau_0(B) < \infty) = 1$, it is immediate from the α-harmonicity that the maximum of $h_0(B)$ is attained at some point in B, this fact often being referred to as discrete maximum principle.

Uniqueness is an immediate consequence of this principle. Consider two solutions h, h' and set $f = k - k'$. Then f and $-f$ provide solutions corresponding to $g(z) = 0,\ z \in B$, and by the maximum principle $f \leq 0,\ -f \leq 0$, hence $h = h'$.

Corollary 1 *Let $B^0 = S, \quad B^k = (B^{k-1})^*,\ k = 1, 2, \ldots$. Then for all $k,\ h_0(B^k)$ is the unique solution of*

$$h(z) = g(z),\ z \in B^k,\ h(z) = \alpha \sum_y p_{zy}\, h(y),\ z \in S \setminus B^k.$$

Furthermore

$$h_1(B^k)(z) = h_0(B^k)(z),\ z \in S \setminus B^k,\ h_1(B^k)(z) = \alpha \sum_y p_{zy}\, h_0(B^k)(y),\ z \in B^k.$$

Proof. The relation between $h_0(B^k)$ and $h_1(B^k)$ is obvious. Hence it is enough to show that $P_z(\tau_0(B^k) < \infty) = 1$ for all z. For this define $F^* = \{z : g(z) = V(z)\}$, where V is the value function of the discounted stopping problem. It is well-known that $\tau_0(F^*)$ is an optimal stopping time and $P_z(\tau_0(F^*) < \infty) = 1$ for all z, see Shiryayev (1978), Chapter 2, Theorem 4. By induction, $B^k \supseteq F^*$ for all k, hence $P_z(\tau_0(B^k) < \infty) = 1$ for all z.

This corollary shows that the iterative step may be performed by solving $h(z) = g(z), z \in B$,$h(z) = \sum_y \alpha\, p_{zy}\, h(y), z \in S$, and then using $\overline{h}(z) = h(z), z \in S \setminus B$, $\overline{h}(z) = \sum_y \alpha\, p_{zy}\, h(y),\ z \in B$ for the comparison with $g(z)$. Using appropriate packages of numerical linear algebra this may be done for large state spaces.

3 Continuous time Markov chains and random discounting

We now treat a continuous time Markov chain $(\widehat{Z}_t)_{t \in [0,\infty)}$ with a finite state space S and look at the discounted optimal stopping problem for

$$\widehat{X}_t = \alpha^t g(\widehat{Z}_t),\ t \in [0, \infty),\ \widehat{X}_\infty = \limsup \widehat{X}_t,$$

assuming $g \geq 0$ for $\alpha < 1$.

It is well-known that an optimal stopping time exists and is of the form

$$\tau = \inf\{t \in [0, \infty) : \widehat{Z}_t \in B\}$$

for some $B \subseteq S$, see Shiryayev (1978). So we only have to look at first entrance times in this stopping problem. Denote the jump times of the chain by $T_0 = 0 < T_1 < T_2 < \ldots$ and the embedded chain by $Z_n = \widehat{Z}_{T_n}$ with transition probabilities $p_{zy},\ z, y \in S$. Given $Z_0 = z_0, \ldots,\ Z_n = z_n$, the waiting times $T_i - T_{i-1},\ i = 1, \ldots, n$, are independent exponentially distributed with parameters $\lambda(z_1), \ldots,\ \lambda(z_n)$. The transition probabilities p_{zy} and the parameters are computed from the Q-matrix of the chain.

Now look at any a.s. finite first entrance time $\widehat{\tau} = \inf\{t : \widehat{Z}_t \in B\}$. Then for $D_n = (B^c)^{n-1} \times B$

$$\begin{aligned}
E\widehat{X}_\tau &= \sum_n E\left(1_{\{\tau=T_n\}}\,\alpha^{T_n}\,g\left(\widehat{Z}_{T_n}\right)\right) \\
&= \sum_n E\left(1_{\{(Z_0,\ldots,Z_n)\in D_n\}}\prod_{i=1}^n \alpha^{T_n-T_{n-1}}\,g\left(Z_n\right)\right) \\
&= \sum_n E\left(1_{\{(Z_0,\ldots,Z_n)\in D_n\}}\,g(Z_n)E\left(\prod_{i=1}^n \alpha^{T_n-T_{n-1}}\mid Z_1,\ldots,Z_n\right)\right) \\
&= \sum_n E\left(1_{\{(Z_0,\ldots,Z_n)\in D_n\}}\,g(Z_n)\prod_{i=0}^{n-1}\alpha(Z_i)\right) \\
&= E\left(g(Z_\tau)\prod_{i=0}^{n-1}\alpha(Z_i)\right),
\end{aligned}$$

where $\tau = \inf\{n : Z_n \in B\}$ and $\alpha(z) = \frac{\lambda(z)}{\lambda(z)-\log\alpha}$ is the moment generating function of an exponential distribution with parameter $\lambda(z)$. So the continuous time problem is equivalent to a discrete time problem with random discounting where $X_n = g(Z_n)\prod_{i=0}^{n-1}\alpha(Z_i)$. The algorithmic step is done in the following way. For $B \subseteq S$ set

$$B^* = \{z : g(z) \geq E_z(g(Z_{\tau_1(B)})\prod_{i=1}^{\tau_1(B)-1}\alpha(Z_i))\}.$$

Again let $B^0 = S,\ B^k = (B^{k-1})^*$ and $F = \bigcap_k B^k$. Under the condition $E_z\,X_{\lim\,\sigma_n} = \lim\,E_z\,X_{\sigma_n}$ for all z and all increasing sequences $(\sigma_n)_n$ of stopping rules, $\tau_0(F)$ again is an optimal stopping time. We give a proof of this result in the Appendix.

Writing as before

$$h_i(B)(z) = E_z\left(g(Z_{\tau_i(B)})\prod_{i=0}^{\tau_i(B)-1}\alpha(Z_i)\right)$$

it follows as in Section 2 that $h_0(B)$ is the unique solution of

$$h(z) = g(z),\ z \in B,\ h(z) = \alpha(z)\sum_y p_{zy}\,k(y),\ z \in S\setminus B.$$

Furthermore

$$h_1(B)(z) = h_0(B)(z),\ z \in S\setminus B,\ h_1(B)(z) = \alpha(z)\sum_y p_{zy}\,h_0(B)(y),\ z \in B.$$

So we may use again numerical linear algebra for FII. We now state and prove the validity of FII in the case of random discounting.

Theorem 1 *Consider a Markovian stopping problem with random discounting where* $g : S \to \mathbb{R},\ \alpha : S \to [0,1]$ *are measurable functions. Let*

$$X_n = g(Z_n)\prod_{i=0}^{n-1}\alpha(Z_i), n = 0,1,\ldots,\ X_\infty = \limsup X_n.$$

Assume that E_zX_τ *exists for all stopping rules* τ, $z \in S$ *and* $E_z\lim_n X_{\sigma_n} = \lim_n E_zX_{\sigma_n}$ *for all* $z \in S$ *and all increasing sequences* (σ_n) *of stopping rules.*
Define $B^k,\ k = 0,1,2,\ldots$ *and* $F = \bigcap_k B^k$ *as in Section 3. Then for all* $z \in S$

(i) $E_zX_{\tau(B^0)} \leq E_zX_{\tau(B^1)} \leq \ldots \uparrow E_zX_{\tau(F)}$,
(ii) $E_zX_{\tau(F)} = V(z)$, *i.e.* $\tau(F)$ *is optimal.*

Proof. The proof uses the basic ideas of the proof of Theorem 1 in Irle (1980) adapted to the Markovian setting with discounting.
(a) Let $B \subseteq S, B^*$ as above. For $\sigma \in \mathcal{T}(B)$ set

$$\sigma^* = \inf\{n \geq \sigma : Z_n \in B^*\}$$

with $\mathcal{T}(B) = \{\tau : Z_\tau \in B \text{ on } \{\tau < \infty\}$. We shall show $E_z X_{\sigma^*} \geq E_z X_\sigma$ from which (i) immediately follows. Let

$$\widehat{\sigma} = \sum_{n<\infty} \tau_{n+1}(B) 1_{\{\sigma=n, Z_n \notin B^*\}} + \sigma 1_{\{Z_\sigma \in B^*\}} + \infty 1_{\{\sigma=\infty\}}.$$

Obviously $\widehat{\sigma} \in \mathcal{T}(B)$. For any $n < \infty$, with an independent copy $(Z_n{}')$ of (Z_n)

$$\begin{aligned}
\int_{\{\sigma=n, Z_n \notin B^*\}} X_\sigma dP_z &= \int_{\{\sigma=n, Z_n \notin B^*\}} g(Z_n) \prod_{i=0}^{n-1} \alpha(Z_i) dP_z \\
&\leq \int_{\{\sigma=n, Z_n \notin B^*\}} \left[E_{Z_n} g\left(Z'_{\tau'_1(B)}\right) \prod_{i=0}^{\tau'_1(B)-1} \alpha(Z'_i) \right] \prod_{i=0}^{n-1} \alpha(Z_i) dP_z \\
&= \int_{\{\sigma=n, Z_n \notin B^*\}} E\left(g(Z_{\tau_{n+1}(B)}) \prod_{i=n}^{\tau_{n+1}(B)-1} \alpha(Z_i) | \mathcal{A}_n\right) \prod_{i=0}^{n-1} \alpha(Z_i) dP_z \\
&= \int_{\{\sigma=n, Z_n \notin B^*\}} g(Z_{\tau_{n+1}(B)}) \prod_{i=0}^{\tau_{n+1}(B)-1} \alpha(Z_i) dP_z \\
&= \int_{\{\sigma=n, Z_n \notin B^*\}} X_{\tau_{n+1}(B)} dP_z.
\end{aligned}$$

This shows $E_z X_{\widehat{\sigma}} \geq E_z X_\sigma$. Next let

$$\sigma_0 = \sigma \text{ and by induction } \sigma_k = \widehat{\sigma}_{k-1},$$

so that $E_z X_\sigma \leq E_z X_{\sigma_k} \leq E_z X_{\sigma_{k+1}} \leq \dots$ for all k. We have $\sigma \leq \sigma_1 \leq \sigma_2 \leq \dots \leq \sigma^*$.

Assume that $\sup \sigma_k(\omega) < \sigma^*(\omega)$ for some ω. Then $\sigma_k(\omega) = \sigma_{k+1}(\omega) < \infty$, for some k, hence $Z_{\sigma_k}(\omega) \in B^*$ and $\sigma_{k+1}(\omega) = \sigma^*(\omega)$. This contradiction implies $\sigma^* = \lim_k \sigma_k$, so that by assumption

$$E_z X_{\sigma^*} = \lim E_z X_{\sigma_k} \geq E_z X_\sigma.$$

(b) We shall now show that for any stopping time σ there exists $\tau \in \mathcal{T}(F)$ such that $E_z X_\tau \geq E_z X_\sigma$. Define

$$\sigma^k = \inf\{n \geq \sigma : Z_n \in B^k\}, \ \tau = \inf\{n \geq \sigma : Z_n \in F\}.$$

Then $\sigma^0 = \sigma \leq \sigma^1 \leq \sigma^2 \leq \dots \leq \tau, \ \sigma^{k+1} = (\sigma^k)^*$. It follows from (a) that

$$E_z X_\sigma \leq E_z X_{\sigma^k} \leq E_z X_{\sigma^{k+1}} \text{ for all } k.$$

Since $F = \bigcap_k B^k$, we obtain $\lim \sigma^k = \tau$, hence by assumption

$$E_z X_\sigma \leq \lim E_z X_{\sigma^k} = E_z X_\tau.$$

(c) Let $\rho, \tau \in \mathcal{T}(F)$ such that $\rho \leq \tau$. We shall show $E_z X_\rho \geq E_z X_\tau$. Let

$$\rho_k = \rho 1_{\{\rho=\tau\}} + \sum_{n<\infty} \tau_{n+1}(B^k) 1_{\{\rho=n<\tau\}}, \ \rho^* = \rho 1_{\{\rho=\tau\}} + \sum_{n<\infty} \tau_{n+1}(F) 1_{\{\rho=n<\tau\}}.$$

Then $\rho \le \rho_k \le \rho_{k+1} \le \rho^* \le \tau,\ \rho^* = \lim \rho_k$. Since $\rho \in \mathcal{T}(F)$ we have for any k

$$\{\rho = n < \tau\} \subseteq \{g(Z_n) \ge E_{Z_n} g(Z'_{\tau'_1(B^k)}) \prod_{i=0}^{\tau'_1(B^k)-1} \alpha(Z'_i)\},$$

implying

$$\begin{aligned}
\int_{\{\rho=n<\tau\}} X_\rho \, dP_z &= \int_{\{\rho=n<z\}} g(Z_n) \prod_{i=1}^{n-1} \alpha(Z_i) \, dP_z \\
&\ge \int_{\{\rho=n<\tau\}} [E_{Z_n} \, g(Z'_{\tau'_1(B^k)}) \prod_{i=0}^{\tau'_1(B^k)-1} \alpha(Z'_i)] \prod_{i=0}^{n-1} \alpha(Z_i) \, dP_z \\
&= \int_{\{\rho=n<\tau\}} g(Z_{\tau_{n+1}(B^k)}) \prod_{i=0}^{\tau_{n+1}(B^k)-1} \alpha(Z_i) \, dP_z = \int_{\{\rho=n<\tau\}} \rho_k \, dP_z,
\end{aligned}$$

hence $E_z X_\rho \ge E_z X_{\rho_k}$. Letting $k \to \infty$ we obtain $E_z X_{\rho^*} \le E_z X_\rho$.

Define $\rho^0 = \rho$, and $\rho^k = (\rho^{k-1})^*$ for $k \ge 1$. Then obviously $\rho \le \rho^k \le \rho^{k+1}$, $\rho_k \in \mathcal{T}(F)$, and $\lim \rho^k = \tau$. It follows

$$E_z X_\rho \ge \lim E_z X_{\rho^k} = E_z X_\tau.$$

(d) Let σ be any stopping time. Then by (b) there exists $\tau \in \mathcal{T}(F)$ such that $E_z X_\sigma \le E_z X_\tau$. By definition $\tau_0(F) \le \tau$, hence by (c)

$$E_z X_{\tau_0(F)} \ge E_z X_\tau \ge E_z X_\sigma.$$

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